Optimization

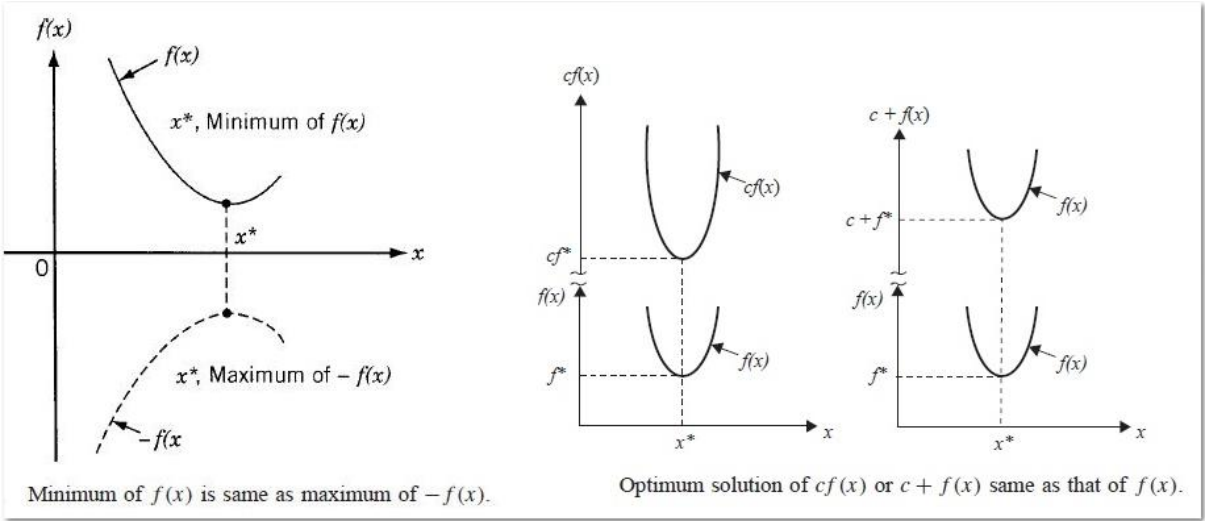
Mathematical Optimization (**Mathematical Programming)** is the selection of a best result, with regard to some criteria. Mathematician have to take many technological and managerial decisions at several stages. The final goal of all such decisions is either to minimize the effort required or to maximize the desired benefit.

التحسين الرياضي (البرمجة الرياضية) هو اختيار أفضل نتيجة وفقا لبعض المعايير. يتعين على عالم الرياضيات اتخاذ العديد من القرارات التكنولوجية والإدارية في عدة مراحل. الهدف النهائي لجميع هذه القرارات هو إما تقليل الجهد المطلوب أو تعظيم الفائدة المرجوة.

Since the effort required or the benefit desired in any practical situation can be expressed as a function of certain decision variables, **optimization** can be defined as the process of finding the conditions that give the maximum or minimum value of a function. It can be seen from figure down that if a point x^* corresponds to the minimum value of function f(x), the same point also corresponds to the maximum value of the negative of the function, $-f(x)$. In addition, the following operations on the objective function will not change the optimum solution x^*:

نظرا لأن الجهد المطلوب أو الفائدة المرغوبة في أي موقف عملي يمكن التعبير عنها كدالة لمتغيرات قرار معينة، يمكن تعريف التحسين على أنه عملية إيجاد الشروط التي تعطي القيمة القصوى أو الدنيا للدالة. يمكن أن نرى من الشكل السفلي أنه إذا كانت النقطة x^* تقابل الحد الأدنى لقيمة الدالة f (x) ، فإن نفس النقطة تتوافق أيضًا مع القيمة القصوى لسالب الدالة $-f(x)$. بالإضافة إلى ذلك ، لن تغير العمليات التالية على دالة الهدف الحل الأمثل:

1. Multiplication (or division) of f(x) by a positive constant c.
2. Addition (or subtraction) of a positive constant c to (or from) f (x).



Minimum of $f(x)$ is same as maximum of $-f(x)$.

Optimum solution of $cf(x)$ or $c + f(x)$ same as that of $f(x)$.

There is no single method available for solving all optimization problems efficiently. Hence a number of optimization methods have been developed for solving different types of optimization problems. The optimum seeking methods are also known as mathematical programming techniques and are generally studied as a part of operations research.

لا توجد طريقة واحدة متاحة لحل جميع مشكلات التحسين بكفاءة. ومن ثم تم تطوير عدد من طرق التحسين لحل أنواع مختلفة من مشاكل التحسين. تعرف طرق البحث المثلى أيضا باسم تقنيات البرمجة الرياضية ويتم دراستها بشكل عام كجزء من أبحاث العمليات.

Optimization can be applied to solve any real life and engineering problems. Some typical applications from different engineering disciplines indicate the wide scope of the subject such as:

1. Design of aircraft and aerospace structures for minimum weight
تصميم هياكل الطائرات ومراكب الفضاء لأدنى وزن
2. Finding the optimal trajectories of space vehicles
إيجاد المسار الأمثل للمركبات الفضائية
3. Design of civil engineering structures such as frames, foundations, bridges, towers, chimneys, and dams for minimum cost.
تصميم الهياكل الهندسية المدنية مثل الهياكل والأساسات والجسور, أبراج ومداخن وسدود بأقل تكلفة

4. Minimum-weight design of structures for earthquake, wind, and other types of random loading
تصميم الوزن الأدنى للهياكل للزلازل والرياح وأنواع أخرى من التحميل العشوائي
5. Design of water resources systems for maximum benefit
تصميم أنظمة الموارد المائية لتحقيق أقصى استفادة

6. Optimal plastic design of structures تصميم بلاستيكي مثالي للهياكل (تراكيب)
7. Optimum design of linkages, cams, gears التصميم الأمثل للوصلات ، الكامات ، التروس

8. Other problems such as Knapsack problem, Assignment problem and Facility Location problem, Set covering/packing/partitioning problem, The travelling salesman (TSP) problem.

In general, optimization is divided into two subfields: discrete optimization and continuous optimization. Optimization problems arise in all quantitative disciplines from computer science and engineering to operations research and economics, and the development of solution methods has been of interest in mathematics for centuries.

Optimization Problem

An optimization or a mathematical programming problem can be stated as follows

$$\text{Find } \mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \text{ which minimizes } f(\mathbf{X})$$

subject to the constraints

$$\begin{aligned} g_j(\mathbf{X}) &\leq 0, \qquad j = 1, 2, \ldots, m \\ l_j(\mathbf{X}) &= 0, \qquad j = 1, 2, \ldots, p \end{aligned} \tag{1.1}$$

where **X** is an n-dimensional vector called the design vector, f (**X**) is termed the objective function, and g_j (**X**) and l_j (**X**) are known as inequality and equality constraints, respectively. The number of variables *n* and the number of constraints *m* and/or *p* need not be related in any way. The problem stated in Eq. (1.1) is called a constrained optimization problem.
Some optimization problems do not involve any constraints and can be stated as

$$\text{Find } \mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix} \text{ which minimizes } f(\mathbf{X}) \tag{1.2}$$

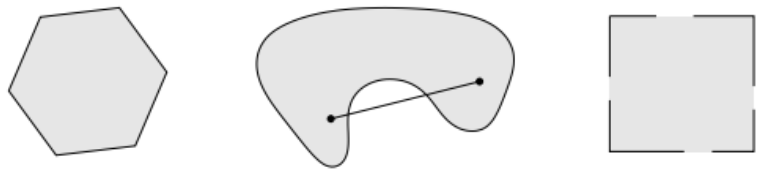
Such problems are called unconstrained optimization problems.

- The classical methods of optimization are useful in finding the optimum solution of continuous and differentiable functions. These methods are analytical and make use of the techniques of differential calculus in locating the optimum points.

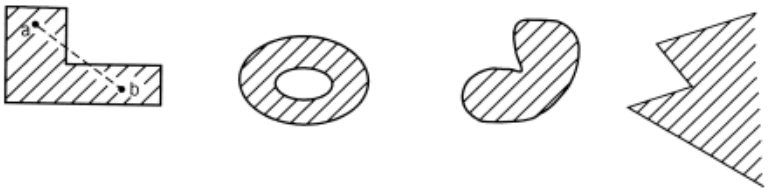
الطرق الكلاسيكية للتحسين مفيدة في إيجاد الحل الأمثل للدوال المستمرة والقابلة للتفاضل. هذه الطرق تحليلية وتستفيد من تقنيات حساب التفاضل في تحديد النقاط المثلى.

Definitions:

Convex sets: A convex set is a collection of points such that if X_1 and X_2 are any two points in the collection, the line segment joining them is also in the collection. A convex set, S, can be defined mathematically as follows: If X_1, $X_2 \in S$, then $X \in S$ where $X = \lambda X_1 + (1 - \lambda) X_2$, $0 \leq \lambda \leq 1$.



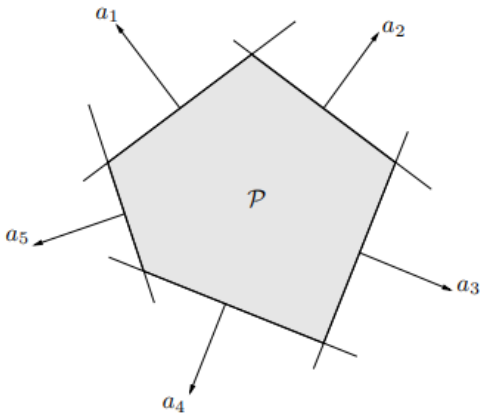
Convex sets.



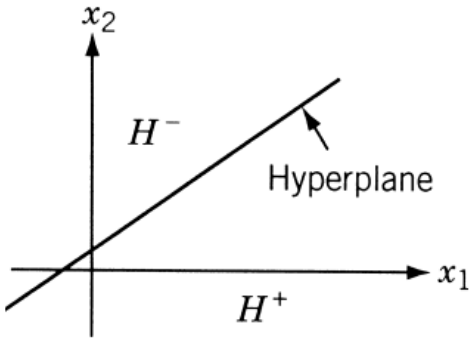
Nonconvex sets.

Some simple convex and nonconvex sets. Left. The hexagon, which includes its boundary (shown darker), is convex. Middle. The kidney shaped set is not convex, since the line segment between the two points in the set shown as dots is not contained in the set. Right. The square contains some boundary points but not others, and is not convex.

Polyhedra: A polyhedron is defined as the solution set of a finite number of linear equalities and inequalities. A polyhedron is thus the intersection of a finite number of half spaces and hyperplanes. It is easily shown that polyhedra are convex sets. A bounded polyhedron is sometimes called a polytope. An example of a polyhedron P (shown shaded) defined as the intersection of five half spaces with outward normal vectors $a_1, \ldots, a_5$.



polytope

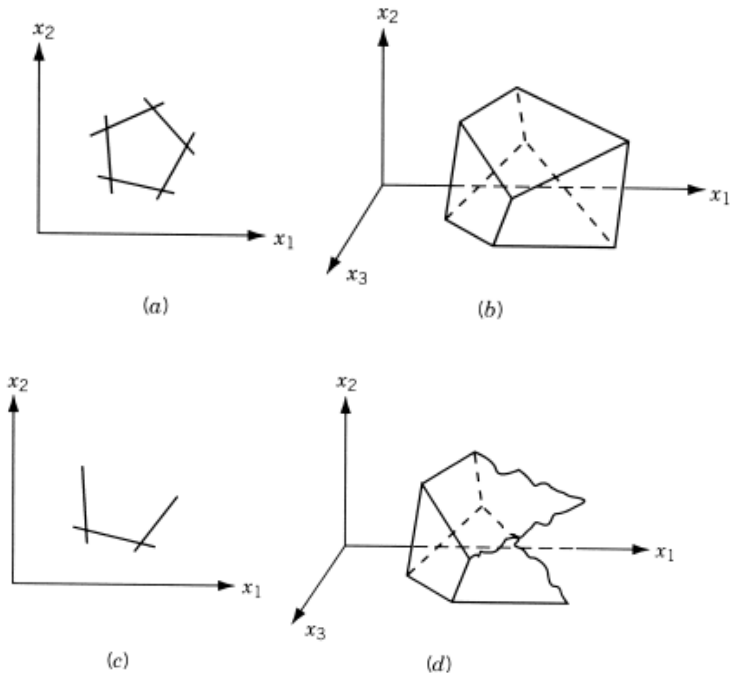


Hyperplane in two dimensions

A convex polyhedron is a set of points common to one or more half-spaces. A convex polyhedron that is bounded is called a convex polytope.

متعدد السطوح المحدب هو مجموعة من النقاط المشتركة في واحد أو أكثر من انصاف الفضاءات.
يسمى متعدد السطوح المحدب الذي يحده بالبوليتوب المحدب

Figures a and b represents convex polytopes in two and three dimensions, and Figures c and d denotes convex polyhedra in two and three dimensions. It can be seen that a convex polygon, shown in Figures a and c, can be considered as the intersection of one or more half-planes.



Vertex or extreme point: It is a point in the convex set that does not lie on a line segment joining two other points of the set. For example, every point on the circumference of a circle and each corner point of a polygon can be called a vertex or extreme point.

Feasible solution. In a linear programming problem, any solution that satisfies the constraints

$$ax = b \qquad (1.3)$$

$x \geq 0$ is called a feasible solution.

Basic solution. A basic solution is one in which (n – m) variables are set equal to zero. It can be obtained by setting (n – m) variables to zero and solving the constraints Eqs. 1.3 simultaneously.

Basis. The collection of variables not set equal to zero to obtain the basic solution is called the basis.

Basic feasible solution. This is a basic solution that satisfies the nonnegativity conditions of

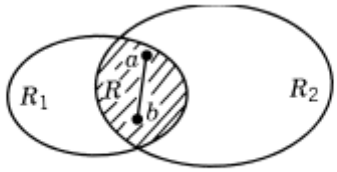
$$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$$

Nondegenerate basic feasible solution. This is a basic feasible solution that has got exactly (m) positive x_i.

Optimal solution. A feasible solution that optimizes the objective function is called an optimal solution.

Optimal basic solution. This is a basic feasible solution for which the objective function is optimal.

Theorem1. The intersection of any number of convex sets is also convex.



Intersection of two convex sets.

Theorem2. The feasible region of a linear programming problem is convex.

Theorem3. Every basic feasible solution is an extreme point of the convex set of feasible solutions.

Operations research

Operations Research (OR) is a branch of mathematics concerned with the application of scientific methods and techniques to decision making problems and with establishing the best or optimal solutions. The beginnings of the subject of operations research can be traced to the early period of World War II. During the war, the British military faced the problem of allocating very scarce and limited resources (such as fighter airplanes, radars, and submarines) to several activities (deployment to numerous targets and destinations).

بحوث العمليات (OR)هي فرع من فروع الرياضيات يهتم بتطبيق الأساليب والتقنيات العلمية على مشاكل صنع القرار ووضع أفضل او أمثل الحلول. يمكن إرجاع بدايات موضوع أبحاث العمليات إلى الفترة المبكرة من الحرب العالمية الثانية. خلال الحرب ، واجه الجيش البريطاني مشكلة تخصيص موارد نادرة للغاية ومحدودة (مثل الطائرات المقاتلة والرادارات والغواصات) للعديد من الأنشطة (الانتشار في العديد من الأهداف والوجهات).

Table Methods of Operations Research

Mathematical programming or optimization techniques	Stochastic process techniques	Statistical methods
Calculus methods	Statistical decision theory	Regression analysis
Calculus of variations	Markov processes	Cluster analysis, pattern recognition
Nonlinear programming	Queueing theory	
Geometric programming	Renewal theory	Design of experiments
Quadratic programming	Simulation methods	Discriminate analysis (factor analysis)
Linear programming	Reliability theory	
Dynamic programming		
Integer programming		
Stochastic programming		
Separable programming		
Multiobjective programming		
Network methods: CPM and PERT		
Game theory		
Modern or nontraditional optimization techniques		
Genetic algorithms		
Simulated annealing		
Ant colony optimization		
Particle swarm optimization		
Neural networks		
Fuzzy optimization		

A **model** is a logical representation of a problem. It shows the relationships between the different variables in the problem.

Approach of Operations Research

1) Definition of the problem.
2) Construction of the model.
3) Solution of the model.
4) Validation of the model.
5) Implementation of the final results.

The main goal: Operations research is often concerned with determining the extreme values of some real-world **objective: the maximum (of profit, performance, or yield) or minimum (of loss, risk, or cost).**

References:

1- Engineering Optimization: Theory and Practice, Fourth Edition Singiresu S. Rao. 2009 by John Wiley & Sons, Inc.
2- Numerical optimization / Jorge Nocedal, Stephen J. Wright. Springer series in operations research, 1999.